Understanding the Orbital Angular Momentum of Light with the aim of implementing logic gates

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I. INTRODUCTION

Light has many degrees of freedom, including polarization, frequency, wavelength, Orbital Angular Momentum (OAM), among others. OAM is a property of light beams with a helical phase structure, first demonstrated by Allen in 1992 while studying the behavior of Laguerre-Gauss beams [1, 2]. Although research on these beams initially progressed slowly due to the demanding generation process, recent advancements in tools such as Spatial Light Modulators (SLM), which enable the manipulation of a beam's transverse phase profile, have facilitated the generation of OAM light beams. When treated quantum mechanically, OAM beams are quantized in integer multiples of $\hbar$ per photon, $\ell\hbar$. This integer value is, in principle, unbounded, meaning that the basis describing the OAM state of a photon can be very large [3]. This property makes OAM particularly useful for information storage and high-dimensional quantum computing.

High-dimensional quantum computing is an emerging field that offers advantages over traditional qubit based quantum computing, such as higher information density per physical entity and more efficient methods for developing quantum algorithms [4]. As mentioned before, OAM of light can be used to encode these high-dimensional quantum states due to its larger basis size.

Just as in classical computing, logic gates form the foundation of information processing in quantum computing. While there is a quantum analogue for basic classical logic gates, these gates must be generalized for states with a basis larger than two basis vectors, leading to the so-called high-dimensional quantum gates [4]. Thus, it is necessary to develop methods for manipulating OAM through optical elements that perform logic gate-like transformations.

As this project is purely theoretical, its development relied on a bibliographic review of the properties of light beams carrying OAM, as well as the necessary calculations for a complete understanding of the physical phenomenon. Additionally, generalizations for the X, Y and Z gates used for the qubit quantum computing are presented. Moreover, since the use of tools is essential for generating and manipulating OAM, a theoretical analysis was conducted on how optical components transform OAM and how these could be implemented to construct high-dimensional logic gates was conducted.

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II. RESULTS

A. Orbital Angular Momentum of Light

Light beams carrying OAM are a result of solving the wave equation for an electromagnetic wave that has the form

$$\tilde{\mathbf{E}}(\mathbf{r},t) = \tilde{E}(\mathbf{r},t)\ \hat{\mathbf{e}}, \tag{1}$$

where $\hat{\mathbf{e}}$ is a polarization unit vector and $\tilde{E}(\mathbf{r},t)$ is the complex field associated with the electric field $\tilde{\mathbf{E}}(\mathbf{r},t)$. The electromagnetic wave is said to propagate in the $\mathbf{k}$ direction and be monochromatic, i.e. to have one and only value for the angular frequency ω. This is, the complex field can be written as

$$\tilde{E}(\mathbf{r},t) = \tilde{E}_0(\mathbf{r})e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} \tag{2}$$

where $\tilde{E}_0(\mathbf{r})$ is a complex amplitude. Due to the field in equation (1) following wave equation, $\tilde{E}$ does too. This is

$$\nabla^2 \tilde{E}(\mathbf{r},t) = \frac{1}{c^2}\frac{\partial^2 \tilde{E}(\mathbf{r},t)}{\partial t^2}$$

when replacing (2) into this last differential equation, one finds that

$$\nabla^2\left[\tilde{E}_0(\mathbf{r})e^{i\mathbf{k}\cdot\mathbf{r}}\right] = -\frac{\omega^2}{c^2}\tilde{E}_0(\mathbf{r})e^{i\mathbf{k}\cdot\mathbf{r}}. \tag{3}$$

By assuming that $\mathbf{k} = k\hat{\mathbf{z}}$, the equation (3) turns into

$$\nabla_\perp^2 \tilde{E}_0(\mathbf{r}) + e^{-ikz}\frac{\partial^2}{\partial z^2}\left[\tilde{E}_0(\mathbf{r})e^{ikz}\right] + k^2\tilde{E}_0(\mathbf{r}) = 0;$$

which, using the paraxial approximation, this is

$$\left|\frac{\partial^2 \tilde{E}_0}{\partial z^2}\right| \ll \left|2k\frac{\partial \tilde{E}_0}{\partial z}\right|,$$

simplifies to

$$\nabla_\perp^2 \tilde{E}_0(\mathbf{r}) + 2ik\frac{\partial \tilde{E}_0(\mathbf{r})}{\partial z} = 0$$

that can be written in cylindrical coordinates as

$$\frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial \tilde{E}_0(\mathbf{r})}{\partial r}\right] + \frac{1}{r^2}\frac{\partial^2 \tilde{E}_0(\mathbf{r})}{\partial \phi^2} + 2ik\frac{\partial \tilde{E}_0(\mathbf{r})}{\partial z} = 0. \tag{4}$$

Now imposing a solution that resembles a propagating Gaussian Beam, according to Kogelnik and Li [5], the solution for the electric field is

$$\tilde{E}_0(\mathbf{r}) = g\left(\frac{r}{w(z)}\right) e^{-i\left(P+\frac{k}{2q}r^2+\ell\phi\right)}$$

here, $g\left(\frac{r}{w(z)}\right)$ is a function depending on r and $w(z)$ the beam waist, that gives the shape of the Laguerre-Gaussian beam; $P(z)$ represents a phase shift due to light beam propagation, such that

$$e^{-iP(z)} = \frac{w_0}{w(z)} e^{-i(2p+\ell+1)\psi(z)}$$

for which $\psi(z) = \arctan\left(\frac{z}{z_R}\right)$, w_0 is the minimum beam waist possible in the beam propagation; and $q(z) = z - iz_R$ represents the Gaussian variation of intensity, where z_R is the Rayleigh length.

After introducing this assumption into the differential equation 4 the family of functions solving this equation is given by the expression [6]

$$\tilde{E}_0(\mathbf{r}) = C_p^\ell \frac{w_0}{w(z)} \left(\frac{\sqrt{2}r}{w(z)}\right)^\ell \times$$
$$L_p^\ell\left(\frac{2r^2}{w^2(z)}\right) e^{-\frac{r^2}{w^2(z)}+ik\frac{r^2}{2R(z)}+i\ell\phi+i\psi(z)}.$$

From this last equation one can observe that the modes of the electric field are quantized by p and ℓ being these numbers associated with radial and angular part, respectively.

When treated quantum-mechanically, the azimuthal part of this electric field $\tilde{E}_0(\mathbf{r})$ can be treated as the position representation of the state $|p,\ell\rangle$, which can be represented via an creation-annihilation approach [7]

$$|p,\ell\rangle = C_p^\ell \left(\hat{a}_+^\dagger \hat{a}_-^\dagger\right)^p \left(\hat{a}_{\mathrm{sgn}(\ell)}^\dagger\right)^{|\ell|} |0\rangle\,, \tag{5}$$

i.e. $\langle r,\phi|p,\ell\rangle = \tilde{E}_0(\mathbf{r})$ [7, 8]. Equation (5) uses $\hat{a}_\pm$ as the creation operators for left- and right-circular modes. In this document, only quantization on OAM of the light beam ℓ will be taken into account, this is: all states will be in basis $\{|\ell\rangle\}_{\ell=0}^{\ell=d-1}$ spanning a d-dimensional Hilbert space.

B. High-dimensional logic gates generalization

When working in a d-dimensional Hilbert space a generalization for Pauli operators is needed, as they represent state transformations analogous to those made by 2-dimensional quantum computer. Let $\hat{X}_d$ and $\hat{Z}_d$ be the d-dimensional generalization for the operators $\hat{\sigma}_x$ and $\hat{\sigma}_z$, respectively. These can be written, in the computational basis, as

$$\hat{X}_d = \sum_{\ell=0}^{d-1} |\ell \oplus 1\rangle \langle \ell|$$
$$\hat{Z}_d = \sum_{\ell=0}^{d-1} \omega^\ell |\ell\rangle \langle \ell|\,,$$

where $\oplus$ is the sum modulo d and ω corresponds to a phase given by $\omega = e^{2\pi i/d}$. By seeing the definition of $\hat{Z}_d$, it is clear that $\{|\ell\rangle\}$ are its eigenstates [9].

Just as with the usual Pauli operators, the properties [10]:

$$\hat{\sigma}_m \hat{\sigma}_n = i \sum_k \epsilon_{mnk} \hat{\sigma}_k + \delta_{mn} \hat{\mathbb{1}} \tag{6}$$
$$\det(\hat{\sigma}_j) = -1 \tag{7}$$
$$\mathrm{tr}(\hat{\sigma}_j) = 0 \tag{8}$$
$$\hat{\sigma}_j^2 = \hat{\mathbb{1}} \tag{9}$$

hold for its d-dimensional counterparts. By looking at the property given by equation (6), an expression for the $\hat{Y}_d$ can be deduced

$$\hat{Y}_d = i\hat{X}_d \hat{Z}_d.$$

This means that, for achieving a $\hat{Y}_d$ it's just necessary to perform $\hat{X}_d$ and $\hat{Y}_d$ operations.

C. Components transforming OAM of light

One of the optical components that is shown to transform OAM of light is the Dove Prism, which acts on the beam adding a phase of 2α when mounted at an angle of α as shown in Figure 1. This means that the exact same operation $\hat{Z}_d$ can be perfomed by using these prisms. When working in a d-dimensional Hilbert space, the Dove Prism must be mounted at a $\frac{\pi}{d}$ angle for it to reproduce the $\hat{Z}_d$.

Additionally, another optical component that transforms OAM is the Spiral Phase Plate (SPP) that uses a birefringent material with an azimuthal dependent thickness and adds an integer value to the topological charge ℓ to any incoming light beam. A schematic of this optical element is shown in Figure 2.

In contrast to $\hat{Z}_d$, $\hat{X}_d$ gate is not as easy to reproduce using optical components. An attempt to reproduce the results theoretically reported by Wang *et al.* [11] has been made. This optical setup is based on what is called a "Parity sorter," which works using a Sagnac interferometer that ensures a phase difference between clockwise and counterclockwise arms. Figure 3 shows the studied setup.

A step-by-step procedure to calculate the state is shown next, which can be followed using Figure 3:

- **Path 1:** Light source for the setup is a Gaussian beam with horizontal polarization $|\psi\rangle = |0\rangle |H\rangle$.
- **Path 2:** Light reflected by the SLM is turns OAM into an arbitrary superposition of the basis $\{|\ell\rangle\}$ and horizontal polarization $|\psi\rangle = \left(\sum_{\ell=0}^{d-1} \alpha_\ell |\ell\rangle\right) |H\rangle$.

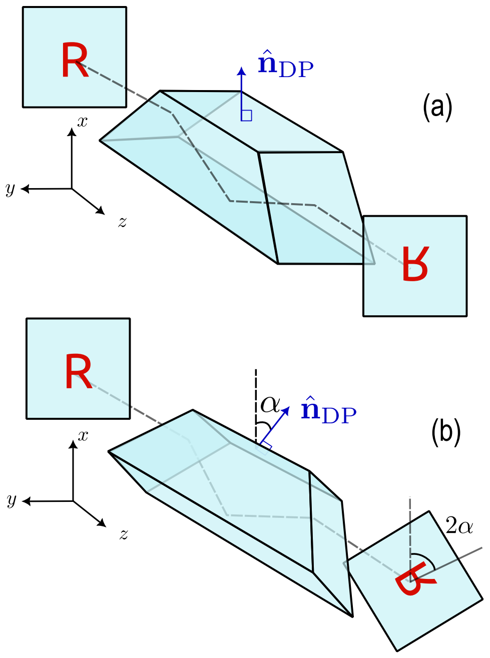

Figure 1. Transformation properties of (a) a unmounted Dove Prism and (b) a Dove Prism mounted at an angle α. Here, $\hat{\mathbf{n}}_{\mathrm{DP}}$ denotes a unitary vector perpendicular to the Dove Prism's top surface. Note that the angle between $\hat{\mathbf{y}}$ and $\hat{\mathbf{n}}_{\mathrm{DP}}$ corresponds to α.

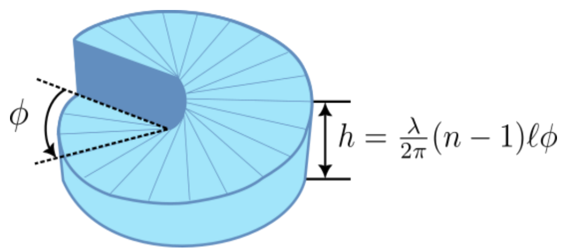

Figure 2. Schematic of a Spiral Phase Plate. The thickness of the birefrigent material depends on the topological charge that wants to be induced on the incoming beam ℓ and the azimuthal angle ϕ. Here, n is the refraction index of the material and λ is the incoming light's wavelength.

- **Path 3:** Light beam passes through a Spiral Phase Plate (SPP) which adds 1 to the topological charge of OAM; the beam still has horizontal polarization $|\psi\rangle = \left(\sum_{\ell=0}^{d-1} \alpha_\ell \, |\ell+1\rangle\right) |H\rangle$.
- **Path 4:** Light passes through a Half-Wave Plate (HWP), this turn the state into $|\psi\rangle = \left(\sum_{\ell=0}^{d-1} \alpha_\ell \, |\ell+1\rangle\right) |D\rangle$.
- **Path 5:** When light passes through the first Polarizing Beam Splitter (PBS) light is put into the state $|\psi\rangle = \left(\sum_{\ell=0}^{d-1} \alpha_\ell \, |\ell+1\rangle\right) |H\rangle$.
- **Path 5':** The state of the clockwise path after passing through the first PBS is set to $|\psi\rangle = -\left(\sum_{\ell=0}^{d-1} \alpha_\ell \, |\ell+1\rangle\right) |V\rangle$.

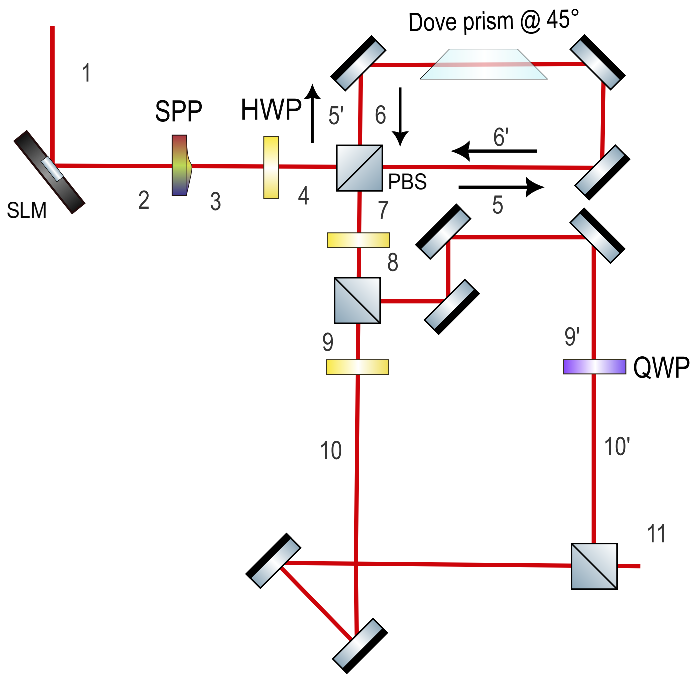

Figure 3. Optical setup for the implementation of a parity sorter. This is supposed to sort between even and odd modes into 9 and 9' paths, respectively, and then recombine them into path 11. Each optical path the light beams passes through is label, primed labels denote that happens at "the same time" as unprimed labels. SLM, SPP, HWP, PBS and QWP are abbreviation for Spatial Light Modulator, Spiral Phase Plate, Half-Wave Plate, Polarizing Beam Splitter and Quarter-Wave Plate, respectively.

- **Path 6:** Light beam, after passing Dove Prim (DP) mounted at 45° turn OAM into $|\psi\rangle = -\left(\sum_{\ell=0}^{3} \alpha_\ell e^{-i\frac{\pi}{2}(\ell+1)} \, |\ell+1\rangle\right) |H\rangle$. Here, d is set to 4 because the DP mounted at 45° degrees performs a $\hat{Z}_4$ gate.
- **Path 6':** Light in the clockwise arm after passing through the DP, the state of light is turned into $|\psi\rangle = \left(\sum_{\ell=0}^{3} \alpha_\ell e^{i\frac{\pi}{2}(\ell+1)} \, |\ell+1\rangle\right) |V\rangle$
- **Path 7:** After passing through the PBS, the state in this path is $|\psi\rangle = \frac{1}{\sqrt{2}}\left(\sum_{\ell=0}^{3} \alpha_\ell e^{-i\frac{\pi}{2}(\ell+1)} \, |\ell+1\rangle\right) |H\rangle + \frac{1}{\sqrt{2}}\left(\sum_{\ell=0}^{3} \alpha_\ell e^{i\frac{\pi}{2}(\ell+1)} \, |\ell+1\rangle\right) |V\rangle$.
- **Path 8:** Light beam passes through a HWP, the state of light is turned into $|\psi\rangle = \sum_{\ell=0}^{3} \cos\left(\frac{\pi}{2}(\ell+1)\right) |\ell+1\rangle \, |V\rangle + \sin\left(\frac{\pi}{2}(\ell+1)\right) |\ell+1\rangle \, |H\rangle$.

After passing through the second PBS, odd modes won't experience any change on path (10), while even modes will undergo a sign flip due to an extra reflection on path (9'), so that in (11) it reproduces an $\hat{X}_d$ gate transformation [11].

III. CONCLUSIONS

Throughout this project, Laguerre-Gaussian beams were described using a wave-equation approach, resulting in an analytical expression for the electric field of Laguerre-Gaussian modes. Additionally, it was shown that these modes can also be represented using annihilation and creation operators for left- and right-circular polarization modes.

Furthermore, a generalization of quantum logic gates to high-dimensional spaces was developed for the fundamental operations $\hat{X}$, $\hat{Y}$, and $\hat{Z}$. Since light is used to encode qudits in the orbital angular momentum (OAM) degree of freedom, optical components capable of transforming this DoF were presented, along with a description of how they affect the OAM states. In particular, the optical setup reported by Wang et al. [11] was analyzed to reproduce the action of the generalized $\hat{X}_d$ gate.

As part of future perspectives, a more detailed analysis of the optical components discussed in this work will be conducted, and the optical setup shown in Figure 3 will be implemented in the laboratory.

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