Final report- Theoretical Project 202520; About Modular Variables in Quantum Information

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In this project, we explored the use of modular variables for encoding adn manipulating quantum information. Firstly, we began by motivating modular variables via Aharonov-Bohm effect. Then, we reproduced the results in Chapter 3 of [1] for the formalism of modular variables and their use for encoding and manipulating quantum information.

I. INTRODUCTION

Consider an electron confined to a region $S \subseteq \mathbb{R}^2$ where $\vec{B} = 0$ but $\vec{A} \neq 0$. Thus, there exists a region $R = \mathbb{R}^2 \setminus S$ with nonzero magnetic field, i.e. a "hole" in the domain. A physical realization is a double slit experiment with a solenoid placed between the slits. Classically, no effect arises, since no force is acting on the particle. Yet, quantum mechanically, we indeed see an effect: the shifting of the interference pattern at a given distance. This is known as the Aharonov-Bohm effect. The effect is depicted in the following picture

In this setup, we consider the wave function of the electron, which, for the sake of the argument, we'll consider a test function $\psi_e \in \mathcal{D}(\mathbb{R})$. Thus, in the case of the Aharonov-Bohm effect, the total wave function will be given by

$$\psi(x) = \psi_e(x + \ell/2) + e^{i\phi}\psi_e(x - \ell/2),$$

where ℓ is the space between slits and $\phi = \frac{e}{ch}\Phi_B$, Φ_B being the magnetic flux through S. Now, consider it's momentum distribution, which is approximately what wel'll see at a far distance from the grating. This will be given by

$$\tilde{\psi}(p) = 2e^{i\phi/2}\tilde{\psi}_e(p)\cos\left(\frac{pl}{2\hbar} - \frac{\phi}{2}\right)$$
$$\implies |\tilde{\psi}(p)|^2 = 4|\tilde{\psi}_e(p)|^2\cos^2\left(\frac{pl}{2\hbar} - \frac{\phi}{2}\right).$$

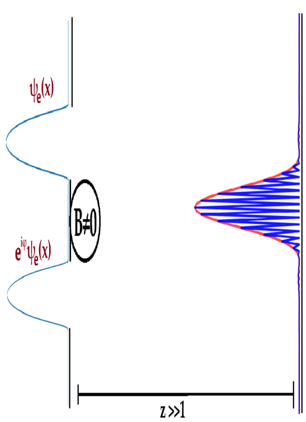

FIG. 1. Illustration of the Aharonov effect. In our case, $z >> 1$ and $\psi_e \in \mathcal{D}(\mathbb{R})$ is a test function.

Thus we see that, evidently, the distribution is dependent on the relative phase. However, we can easily see that since

$$|\psi(x)|^2 = |\psi_e(x)|^2 + |\psi_e(x+\ell/2)|^2 + 2\mathrm{Re}(e^{i\phi}\psi_e(x)\psi_e(x+\ell/2)),$$

then for any $n \in \mathbb{N}$, $\langle x^n \rangle = 0$. We can also see that because of the properties of test functions, namely, that their derivative its also a test function with the same support, for any $m \in \mathbb{N}$, $\langle p^m \rangle = 0$ as well. Thus, we cannot reconstruct the moment distribution, which is the same as the position distribution after a screen at large z, using the any of the moments since they're not sensible to the relative phase. This is a reflect that the characteristic function is not analytic is our case.

So we are in problems. Let us consider the case of the translation in position by ℓ. Thus

$$\langle e^{i\hat{p}\ell/\hbar} \rangle = \langle \psi | \hat{T}_x(\ell) | \psi \rangle = e^{-i\phi}/2.$$

So, we have just found a quantity which is sensible to the relative phase. Moreover, we see that

$$\frac{d}{dt}e^{\frac{i}{\hbar}\hat{p}l} = \frac{i}{\hbar}\left[\hat{H}, e^{\frac{i}{\hbar}\hat{p}l}\right] = \frac{i}{\hbar}(V(\hat{x}) - V(\hat{x}+l))e^{\frac{i}{\hbar}\hat{p}l}.$$

Thus, in the case in which the potential is periodic, the translation operator is a movement constant. The same thing happens for $\hat{T}_p(h/\ell)$. This motivates us to study the following kind of operators

$$f(\hat{x}) = \sum_{n\in\mathbb{Z}} f_n \hat{T}_p(n\alpha) \quad , \quad g(\hat{x}) = \sum_{n\in\mathbb{Z}} g_n \hat{T}_x(n\beta). \tag{1}$$

We notice that this operators are functions of the displacement operators and that they are periodic.

II. MODULAR VARIABLES FORMALISM

Let us examine the commutation relations between this operators. But first, let us examine the commutation

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relations between the momentum and position displacement operators. Notice that

$$[T_p(\alpha), T_x(\beta)] = T_p(\alpha)T_x(\beta) - T_x(\beta)T_p(\alpha)T_x^\dagger(\beta)T_x(\alpha).$$

Besides

$$\begin{aligned} T_x(\beta)T_p(\alpha)T_x^\dagger(\beta)|x\rangle &= T_x(\beta)T_p(\alpha)|x+\beta\rangle \\ &= T_x(\beta)e^{i\alpha(x+\beta)/\hbar}|x+\beta\rangle \\ &= e^{i\alpha\beta/\hbar}e^{i\alpha x/\hbar}T_x(\beta)|x+\beta\rangle \\ &= e^{i\alpha\beta/\hbar}T_p(\alpha)|x\rangle. \end{aligned}$$

This leads us to the conclusion that

$$[T_p(\alpha), T_x(\beta)] = (1 - e^{i\alpha\beta/\hbar})T_p(\alpha)T_x(\beta).$$

And thus, our displacement operators commute if and only if there exist an $n \in \mathbb{N}$ such that

$$\alpha\beta = nh. \tag{2}$$

This relation is general for two periodic operators (i.e. linear combinations of displacement operators), with period α and β respectively. That is, this relation applies to operators in 1. In this fashion, let us consider the following partition for momentum and position operators [2]:

$$\begin{aligned} \hat{x} &= \hat{N}\ell + \hat{\bar{x}}, \\ \hat{p} &= \hat{M}\frac{\ell}{h} + \hat{\bar{p}}, \end{aligned}$$

in which $\ell \in \mathbb{R}^+$, $\hat{\bar{x}} = (\hat{x} - x_\ell)\text{mod}(\ell) + x_\ell$, $\hat{\bar{p}} = (\hat{p} - p_\ell)\text{mod}(\ell/h) + p_\ell$, and $\hat{N}, \hat{M}$ are operators that take integer eigenvalues. We call $\hat{\bar{x}}$ and $\hat{\bar{p}}$ the modular position and momentum operators. It is relevant to say that the eigenvalues of both modular position and momentum are in intervals given by $\bar{x} \in [x_\ell, x_\ell + \ell]$ and $[p_l, p_l + \hbar/\ell]$, where x_ℓ, p_ℓ are real constant which we can choose. For the future calculations, we will choose $x_\ell = -\ell/2$ and $p_\ell = -\hbar/2\ell$. They have great and profound physical meaning (see [2]).But not only that, they are very useful for our purposes.

Notice that our modular position and momentum operators, by equation **??**, commute. Thus, we can find a common eigenbasis $|\bar{x}, \bar{p}\rangle$ our modular eigenstates. Indeed, this will we our main tool for encoding quantum information. Let us find analytical expressions for this eigenbasis. Moreover, it is easy to see that

$$T_p(\alpha) = e^{i\alpha\hat{p}/\hbar} = e^{i\alpha\hat{\bar{p}}}, \tag{3}$$

since $\hat{M}$ only takes integer eigenvalues. Same thing happens with $T_x(\beta)$. Not only that, since

$$\hat{\bar{x}}|\bar{x}, \bar{p}\rangle = \bar{x}|\bar{x}, \bar{p}\rangle \quad , \quad \hat{\bar{p}} = \bar{p}|\bar{x}, \bar{p}\rangle,$$

it follows that [1]

$$e^{i\hat{\bar{x}}/\ell\hbar}|\bar{x}, \bar{p}\rangle = e^{i\bar{x}/\ell\hbar}|\bar{x}, \bar{p}\rangle, \tag{4}$$

$$e^{-i\ell\hat{\bar{p}}/\hbar}|\bar{x}, \bar{p}\rangle = e^{-i\ell\bar{p}/\hbar}|\bar{x}, \bar{p}\rangle. \tag{5}$$

Then, we can check from equation (3), using 3, that for $x \in \mathbb{R}$

$$\begin{aligned} \langle x|e^{i\hat{x}/\hbar\ell}|\bar{x}, \bar{p}\rangle &= e^{i\hat{x}/\hbar\ell}|\bar{x}, \bar{p}\rangle \\ \implies e^{i/\ell\hbar(x-\bar{x})}\langle x|\bar{x}, \bar{p}\rangle &= \langle x|\bar{x}, \bar{p}\rangle \\ \implies x - \bar{x} &= m\ell\hbar \\ \implies x &= \bar{x} + m\ell\hbar \quad , \quad m \in \mathbb{Z}. \end{aligned}$$

Thus, $\langle x|\bar{x}, \bar{p}\rangle$ has a discrete support. Now, notice that equation (4) implies that

$$\begin{aligned} \langle x|e^{-i\hat{p}\ell/\hbar}|\bar{x}, \bar{p}\rangle &= e^{-i\bar{p}\ell/\hbar}\langle x|\bar{x}, \bar{p}\rangle \\ \implies \langle x+\ell|\bar{x}, \bar{p}\rangle &= e^{-i\bar{p}\ell/\hbar}\langle x|\bar{x}, \bar{p}\rangle. \end{aligned}$$

If we combine both results we can see that

$$\langle x|\bar{x}, \bar{p}\rangle = \sqrt{\frac{\ell}{2\pi}}\sum_{m\in\mathbb{Z}} e^{-i\bar{p}m}\delta(x - m\ell\hbar - \bar{x}).$$

And it easily follows that in the position basis

$$|\bar{x}, \bar{p}\rangle = \sqrt{\frac{\ell}{2\pi}}\sum_{m\in\mathbb{Z}} e^{-i\bar{p}m}|\bar{x} + m\ell\hbar\rangle.$$

And thus, we can write any state $|\psi\rangle$ as

$$|\psi\rangle = \int_{-\ell/2}^{\ell/2}\int_{-\hbar/2\ell}^{\hbar/2\ell} d\bar{x}d\bar{p}\ \Psi(\bar{x}, \bar{p})|\bar{x}, \bar{p}\rangle,$$

with

$$\Psi(\bar{x}, \bar{p}) = \sqrt{\frac{\ell}{2\pi}}\sum_{m\in\mathbb{Z}} e^{-i\bar{p}m\ell}\langle\bar{x} + m\ell\hbar|\psi\rangle.$$

Furthermore, when analyzing the free-particle time evolution of these modular eigenstates, one observes a characteristic shear of the phase-space lattice of the form $(x, p) \mapsto (x + tp/m, p)$. This deformation closely resembles the Talbot effect, in which self-imaging and fractional revivals emerge from periodic structures under free evolution. The analogy highlights the rich interference phenomena naturally encoded in modular variables and their relevance for phase-space based quantum information protocols.

Moreover, we can write any operator A with our modular formalism like

$$A = \int_{-\ell/2}^{\ell/2} d\bar{x}d\bar{x}'\int_{-\hbar/2\ell}^{\hbar/2\ell} d\bar{p}d\bar{p}'\langle\bar{x}, \bar{p}|A|\bar{x}', \bar{p}'\rangle|\bar{x}, \bar{p}\rangle\langle\bar{x}', \bar{p}'|, \tag{6}$$

where

$$\langle\bar{x}, \bar{p}|A|\bar{x}', \bar{p}'\rangle = \frac{\ell}{2\pi}\sum_{r,s\in\mathbb{Z}} e^{i(\bar{p}'s - \bar{p}r)\ell}\langle\bar{x} + s\ell|A|\bar{x}' + r\ell\rangle.$$

This expressions will be really useful later on when we define the logical operations in our modular formalism for quantum computation.

III. DEFINITION ON LOGICAL STATES

Now, we wish to use the formalism we've developed for encoding qubits in phase space. For this purpose, we will make a partition of the box we are considering. That is, we will split the integration domain in two. In particular, we will make the partition with respect to the position. With this in mind, we will consider now $x_\ell = -\ell/4$. In this case, it is easy to see that for an arbitrary state $|\psi\rangle$

$$|\psi\rangle = \int_{-\ell/4}^{\ell/4} d\bar{x} \int_{-\hbar/2\ell}^{\hbar/2\ell} \Psi(\bar{x},\bar{p})|\bar{x},\bar{p}\rangle + \Psi(\bar{x}+\ell/2,\bar{p})|\bar{x}+\ell/2,\bar{p}\rangle.$$

Now, if $a, b \in \mathbb{C}$, we can always find $\theta, \phi \in \mathbb{R}$ and $f \in \mathbb{C}$ such that

$$a = f \sin\left(\frac{\theta}{2}\right) \tag{7}$$

$$b = f e^{i\phi} \cos\left(\frac{\theta}{2}\right) \tag{8}$$

$$|f| = \sqrt{|a|^2 + |b|^2}. \tag{9}$$

Thus, it follows that there exist functions $f(\bar{x},\bar{p})$, $\theta(\bar{x},\bar{p}), \phi(\bar{x},\bar{p})$ such that

$$\Psi(\bar{x},\bar{p}) = f(\bar{x},\bar{p}) \cos\left(\frac{\theta(\bar{x},\bar{p})}{2}\right)$$

$$\Psi(\bar{x}+\ell/2,\bar{p}) = f(\bar{x},\bar{p}) e^{i\phi(\bar{x},\bar{p})} \sin\left(\frac{\theta(\bar{x},\bar{p})}{2}\right).$$

Now since we are considering a torus, we assume $\theta(\bar{x},\bar{p}) = \theta$ and $\phi(\bar{x},\bar{p}) = \phi$ are constant for a given $|\psi\rangle$. In this case, we have that

$$|\psi\rangle = \cos\left(\frac{\theta}{2}\right)|0_L\rangle + \sin\left(\frac{\theta}{2}\right) e^{i\phi}|1_L\rangle, \tag{10}$$

with

$$|0_L\rangle = \int_{-\ell/4}^{\ell/4} d\bar{x} \int_{-\hbar/2\ell}^{\hbar/2\ell} d\bar{p} f(\bar{x},\bar{p})|\bar{x},\bar{p}\rangle, \tag{11}$$

$$|1_L\rangle = \int_{-\ell/4}^{\ell/4} d\bar{x} \int_{-\hbar/2\ell}^{\hbar/2\ell} d\bar{p} f(\bar{x},\bar{p})|\bar{x}+\ell/2,\bar{p}\rangle. \tag{12}$$

In equations (6) and (7) the function $f(\bar{x},\bar{p})$ is arbitrary as long as it arises form a properly defined modular wave function $\Psi(\bar{x},\bar{p})$. Equation (5) reflects a dichotomization of the Hilbert space and defines a general way of encoding quantum information in a continuous variable system. The situation is depicted in 2

In particular, we can choose $f(\bar{x},\bar{p})$ to be (in the modular box domain) a two variable Gaussian spike with widths Δ and κ in modular position and momentum [1]. In particular, if we choose the limit in which $\Delta, \kappa \to 0$, we arrive to the usual GKP codewords [3]

$$|0\rangle_{\text{GKP}} = \sum_{m\in} \left|2\sqrt{\pi}m\right\rangle_x \tag{13}$$

$$|1\rangle_{\text{GKP}} = \sum_{m\in} \left|(2m+1)\sqrt{\pi}\right\rangle_x. \tag{14}$$

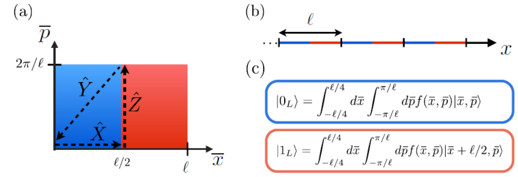

FIG. 2. a) Visual representation of the partition of phase space and the partition of the modular box in which we are working. The arrows represent the Pauli operations $\hat{X}, \hat{Y}$ and $\hat{Z}$. b) Visual representation of the partition seen in the position representantion. c) Logical states which arise from this partition. Taken from [1].

to low error probability.

A physical realization of this is procedure is by using a harmonic oscillator and encoding the GKP codewords in coherent states.

Even though our modular variable formalism is beautiful and relatively easy to manipulate, as everything is life, there is a price that has to be paid. And that price is that modular eigenstates are non-physical since they're not normalizable (they're not in the Hilbert space we're considering, $L_2(\mathbb{C})$). That is, our GKP codespace is non-physical. A way of solving this problem is by instead of using delta peaks, we use a Gaussian spike of width Δ and multiplying them by a Gaussian envelope of width$1/\kappa$ [1], such that

$${}_x\langle x|\tilde{0}\rangle_{\text{GKP}} \sim \frac{N}{(\pi\Delta^2)^{1/4}} e^{-(x\kappa)^2/2} \sum_{m\in\mathbb{Z}} e^{-(x-m\ell)^2/2\Delta^2},$$

where $|\tilde{0}\rangle_{\text{GKP}}$ is our aprroximate codeword and N is a normalization constant. A similar thing happens with $|\tilde{1}\rangle_{\text{GKP}}$. In the limit is which $\Delta \to 0$ and $\kappa \to \infty$ we get our idealized codewords. Another and important approximation is given by introducing small weighted displacements to 13 and 14 [3]:

$$|\tilde{i}\rangle_{GKP} = \int_{\mathbb{R}} dudv \eta(u,v) e^{iuv/2} T_x(u) T_p(v) |i\rangle_{\text{GKP}}, \tag{15}$$

where $i = 0, 1$.

IV. LOGICAL OPERATIONS IN THE MODULAR LOGICAL SPACE

In order to define the Pauli operations in our formalism, let us first define the *displacement operator*, which is given by

$$D(\mu,\nu) = e^{i\nu\hat{x} - i\mu\hat{p}}. \tag{16}$$

And, if one uses 6, one can show that (see [1])

$$D(\mu,\nu) = e^{-i\frac{\mu\nu}{2}} \int_{-\ell/4}^{3\ell/4} d\bar{x} \int_{-h/2\ell}^{h/2\ell} d\bar{p}\ e^{\tau(\bar{x},\bar{p})} |\overline{x+\mu}, \overline{p+\nu}\rangle\langle\bar{x},\bar{p}|, \tag{17}$$

with $\tau(\bar{x}, \bar{p}) = i(\bar{p} + \nu)(\overline{\bar{x} + \mu}) - i\bar{p}(\bar{x} + \nu)$. With this operator is easy to define X and Z as $D(0, h/\ell)$ and $D(\ell/2, 0)$. Similarly, we define $Y = D^{\dagger}(\ell/2, h/\ell)$. We notice that even tough this operators are analogous to the usual Pauli matrices, the big problem is that they're not selfadjoint. However, we can deal with this problem by restricting ourselves to the GKP codespace we described in the last section.

V. CONCLUSIONS

Throughout this project, we achieved several objectives. First, we successfully motivated the introduction of modular variables by connecting them to physically meaningful scenarios such as the Aharonov–Bohm effect. Second, we demonstrated their potential as a resource for encoding and manipulating quantum information, emphasizing how modular degrees of freedom naturally support grid-like structures relevant for fault-tolerant architectures. Finally, future work includes developing numerical simulations aimed at implementing GKP-like encoding schemes using platforms such as Qiskit or PennyLane, in order to explore practical realizations of modular-variable quantum information processing. An additional point worth emphasizing is the importance of analyzing the dynamical behavior of modular eigenstates under free evolution, since their shearing and revival structure is fundamental for understanding phase–space encodings.

[1] A. Ketterer, *Modular variables in quantum information*, Ph.D. thesis, Université Paris Diderot (Octuber, 2016).

[2] Y. Aharonov, H. Pendleton, and A. Petersen, Int. J. Theor. Phys. **2**, 213 (1969).

[3] A. L. Grimsmo and S. Puri, PRX Quantum **2**, 020101 (2021).